Available online at www.sciencedirect.com

SCIENCE DIRECT®

ELSEVIER

Linear Algebra and its Applications 371 (2003) 369–375

LINEAR ALGEBRA AND ITS APPLICATIONS

www.elsevier.com/locate/laa

On weakly unitarily invariant norm and the Aluthge transformation ☆

K. Okubo

Mathematics Laboratory, Hokkaido University of Education, Sapporo College, 5-3-1-5 Ainosato Kitaku, Sapporo 002-8502, Japan

Received 15 January 2003; accepted 11 February 2003

Submitted by R.A. Brualdi

Abstract

In this paper we show that

$$|||f(P^{\lambda}UP^{1-\lambda})||| \leqslant \max\left\{|||f(T)|||,\ |||U^{*}\cdot f(T)\cdot U + f(0)(I-U^{*}U)|||\right\},$$

where $T \in \mathscr{B}(\mathscr{H})$, $|||\cdot|||$ is a semi-norm on $\mathscr{B}(\mathscr{H})$ which satisfies some conditions, $T = UP$ (polar decomposition), $0 \leqslant \lambda \leqslant 1$ and f is a polynomial. As a consequence of this fact, we will show that some semi-norms $|||\cdot|||$ including the ρ-radii $(0 < \rho \leqslant 2)$ satisfy the inequality $|||f(P^{\lambda}UP^{1-\lambda})||| \leqslant |||f(T)|||$. We also give some related results.

AMS classification: 15A60; 47A12; 47A63

Keywords: λ-Aluthge transformation; Aluthge transformation; Weakly unitarily invariant norm; Operator radii; Numerical range

1. Introduction

We denote by $\mathscr{B}(\mathscr{H})$ the algebra of all bounded linear operators on a Hilbert space $\mathscr{H}$. Let $T \in \mathscr{B}(\mathscr{H})$ and $T = UP$ be a polar decomposition of T. Here P is positive semi-definite and U is partial isometry satisfying $U^{*}UP = P$. P is determined uniquely by $P = |T|$, but usually U is not unique for the case of $\ker(T) \neq \{0\}$ and $\ker(T^{*}) \neq \{0\}$. For $0 < \lambda < 1$, we define the λ-*Aluthge transformation* of T

☆ Supported by Grant-Aid for Scietific Research 13640146.

E-mail address: okubo@sap.hokkyodai.ac.jp (K. Okubo).

0024-3795/$ - see front matter © 2003 Published by Elsevier Inc.
doi:10.1016/S0024-3795(03)00485-3

by $P^{\lambda}UP^{1-\lambda}$. The λ-Aluthge transformation of T is determined uniquely without depending on any choice of U. In particular, for $\lambda = 1/2$, $\tilde{T} := P^{1/2}U P^{1/2}$ is called the *Aluthge transformation* of T (see [1]). Yamazaki [7] proved that the numerical range $W(\tilde{T})$ of the Aluthge transformation $\tilde{T}$ of T satisfies the inclusion

$$\overline{W(\tilde{T})} \subset \overline{W(T)}, \tag{1}$$

if T admits a decomposition $T = UP$ for a unitary operator U, and Wu [8] generalized this result to the general case. The numerical radius $w(T)$ of T is defined by

$$w(T) = \sup\{|z| : z \in W(T)\}. \tag{2}$$

Eq. (1) shows

$$w(\tilde{T}) \leqslant w(T). \tag{3}$$

For $\rho > 0$, an operator $T \in \mathscr{B}(\mathscr{H})$ is called a ρ-contraction if there is a Hilbert space $\mathscr{K} \supset \mathscr{H}$ and a unitary operator V on $\mathscr{K}$ such that

$$T^n = \rho QV^n|_{\mathscr{H}} \quad (n = 1, 2, \ldots). \tag{4}$$

Here Q is the orthogonal projection from $\mathscr{K}$ to $\mathscr{H}$. A ρ-contraction is firstly considered by Nagy and Foiaş [6], and Holbrook defined ρ-radius $w_\rho(T)$ of T by

$$w_\rho(T) = \inf\left\{r > 0 : \frac{1}{r}T \text{ is a } \rho - \text{contraction}\right\}. \tag{5}$$

It is known that for $0 < \rho \leqslant 2$, $w_\rho(\cdot)$ is a norm on $\mathscr{B}(\mathscr{H})$ but for $2 < \rho < \infty$ is only a quasi-norm. Each ρ-radius has weakly unitary invariance in the sense

$$w_\rho(VTV^*) = w_\rho(T) \quad (V : \text{unitary}). \tag{6}$$

Moreover ρ-radii have the properties:

$$w_1(T) = \|T\|,$$

where $\|\cdot\|$ is the Hilbert space operator norm and

$$w_2(T) = w(T)$$

and

$$w_\infty(T) := \lim_{\rho\to\infty} w_\rho(T) = r(T),$$

where $r(\cdot)$ is the spectral radius, that is, $r(T) = \max\{|\lambda| : \lambda \in \sigma(T)\}$. Here $\sigma(T)$ is the spectrum of T.

For references of ρ-radius see [4,6].

In this paper we treat the semi-norm $|||\cdot|||$ on $\mathscr{B}(\mathscr{H})$ satisfying the following two conditions:

$$\exists\gamma, \quad |||X||| \leqslant \gamma\|X\| \quad (X \in \mathscr{B}(\mathscr{H})) \tag{7}$$

and

$$|||S^*XS||| \leqslant \|S\|^2 \cdot |||X||| \quad (X, S \in \mathscr{B}(\mathscr{H})). \tag{8}$$

Moreover, sometimes we add the following conditions:

$$|||Q||| = 1 \quad \text{for any orthogonal projection } Q \neq 0 \tag{9}$$

and

$$\begin{aligned} &XQ = QXQ : \text{orthogonal projeciton} \\ &\quad \Longrightarrow |||X||| = \max\{|||XQ|||,\ |||X(1-Q)|||\}. \end{aligned} \tag{10}$$

If $|||\cdot|||$ is equivalent to the operator norm, that is, there is a positive number γ' such that

$$\gamma'|||X||| \geqslant \|X\| \quad (X \in \mathscr{B}(\mathscr{H})),$$

or more naturally if

$$r(X) \leqslant |||X||| \quad (X \in \mathscr{B}(\mathscr{H}))$$

then the condition (9) is satisfied. For $0 < \rho \leqslant 2$, the ρ-radius $w_\rho(\cdot)$ is an example of a norm satisfies (7)–(10).

2. Consequences

In this paper we will give the generalization of inequalitiy (3) to the λ-Aluthge transformation and some weakly unitarily invariant semi-norms. The following inequality is the main result in this paper.

Theorem 1. *Let $T \in \mathscr{B}(\mathscr{H})$, $T = UP$ be a polar decomposition and $|||\cdot|||$ be a semi-norm on $\mathscr{B}(\mathscr{H})$ which satisfies conditions* (7) *and* (8). *Then for $0 \leqslant \lambda \leqslant 1$ and for any polynomial f, we have the inequality*

$$|||f(P^\lambda UP^{1-\lambda})||| \leqslant \max\left\{|||f(T)|||,\ |||U^* \cdot f(T) \cdot U + f(0)(I - U^*U)|||\right\}.$$

Proof. Denote $f(z) = f(0) + g(z)z$. Here $g(z)$ is also a polynomial. We define P_n $(n = 1, 2, \ldots)$ as the following:

$$P_n = \begin{cases} P & (P \text{ is invertible}), \\ P + \frac{1}{n} & (P \text{ is not invertible}). \end{cases}$$

Then P_n in invertible positive definite and

$$0 \leqslant P_n^\lambda - P^\lambda \leqslant \frac{1}{n^\lambda} \quad (n = 1, 2, \ldots;\ 0 < \lambda \leqslant 1).$$

Therefore in the sense of uniform convergence we have

$$\begin{aligned} f(P^\lambda UP^{1-\lambda}) &= \lim_{n\to\infty} f(P_n^\lambda UP_n^{1-\lambda}), \\ f(T) &= \lim_{n\to\infty} f(UP_n). \end{aligned}$$

We consider the following operator-valued analytic function $\varphi_n(z)$ $(n = 1, 2, \ldots)$ on the strip $\mathscr{D} = \{z \in \mathscr{C} : -1/2 \leqslant \mathrm{Re}(z) \leqslant 1/2\}$ in the complex plane $\mathscr{C}$:

$$\varphi_n(z) \equiv f(0) + P_n^{(1/2)-z} \cdot g(UP_n) \cdot UP_n^{(1/2)+z}.$$

We will estimate the norm of $\varphi_n(z)$ on the boundary of $\mathscr{D}$. We have

$$\begin{aligned}\varphi_n\left(\frac{1}{2} + \mathrm{i}t\right) &= f(0) + P_n^{-\mathrm{i}t} \cdot g(UP_n)UP_n \cdot P_n^{\mathrm{i}t} \\ &= P_n^{-\mathrm{i}t} \cdot f(UP_n) \cdot P_n^{\mathrm{i}t}\end{aligned}$$

and

$$\begin{aligned}\varphi_n\left(-\frac{1}{2} + \mathrm{i}t\right) &= f(0) + P_n^{-\mathrm{i}t} P_n \cdot g(UP_n)U \cdot P_n^{\mathrm{i}t} \\ &= P_n^{-\mathrm{i}t}\{f(0) + P_n \cdot g(UP_n)U\}P_n^{\mathrm{i}t}.\end{aligned}$$

Since $P_n^{\pm\mathrm{i}t}$ is unitary, it follows from (8) that

$$\sup_{t\in\mathscr{R}} |||\varphi_n\left(\frac{1}{2} + \mathrm{i}t\right)||| \leqslant |||f(UP_n)|||,$$

$$\sup_{t\in\mathscr{R}} |||\varphi_n\left(-\frac{1}{2} + \mathrm{i}t\right)||| \leqslant |||f(0) + P_n \cdot g(UP_n)U|||.$$

Therefore the three-line theorem (cf. [3, pp. 136–137]) implies that

$$\begin{aligned}|||f(P_n^{\lambda}UP_n^{1-\lambda})||| &= |||\varphi_n\left(\frac{1}{2} - \lambda\right)||| \\ &\leqslant \max\{|||f(UP_n)|||,\ |||f(0) + P_n \cdot g(UP_n)U|||\}.\end{aligned}$$

Taking a limit, we have by (7)

$$|||f(P^{\lambda}UP^{1-\lambda})||| \leqslant \max\{|||f(T)|||,\ |||f(0) + P \cdot g(T)U|||\}. \tag{11}$$

Since $U^*UP = P$,

$$\begin{aligned}f(0) + P \cdot g(T)U &= U^*\{f(0) + UP \cdot g(T)\}U + f(0)(I - U^*U) \\ &= U^* \cdot f(T) \cdot U + f(0)(I - U^*U).\end{aligned}$$

Hence we have

$$|||f(0) + P \cdot g(T)U||| = |||U^* \cdot f(T) \cdot U + f(0)(I - U^*U)||| \tag{12}$$

and by (11) and (12) Theorem 1 is proven. $\square$

Corollary 2. *Let T, U, P, $|||\cdot|||$, f, λ be the same as in Theorem* 1 *and at least one of the following conditions be satisfied*:

(i) $f(0) = 0$,

(ii) $\dim(\ker(T)) \leqslant \dim(\ker(T^*))$,
(iii) $|||\cdot|||$ *satisfies conditions* (9) *and* (10).

Then we have

$$|||f(P^\lambda U P^{1-\lambda})||| \leqslant |||f(T)|||.$$

Proof. In the case of (i) or (ii), we can easily show that

$$|||U^* \cdot f(T) \cdot U + f(0)(I - U^*U)||| = |||U^* \cdot f(T) \cdot U||| \leqslant |||f(T)|||.$$

In the case of (iii) and if U is not an isometry, using the conditions (9) and (10) with $Q = U^*U$, we derive

$$\begin{aligned} |||U^* \cdot f(T) \cdot U + f(0)(I - U^*U)||| &= \max\left\{|||U^* \cdot f(T) \cdot U|||,\ |f(0)|\right\} \\ &\leqslant \max\{|||f(T)|||,\ |f(0)|\}. \end{aligned} \tag{13}$$

On the other hand, since $PU^*U = P$, we obtain

$$f(T)U^*U = (f(0) + g(T)T)U^*U = f(0)U^*U + g(T)T$$

and hence

$$\begin{aligned} |f(0)| \cdot |||I - U^*U||| &= |||(I - U^*U) \cdot f(T) \cdot (I - U^*U)||| \\ &\leqslant |||f(T)|||. \end{aligned} \tag{14}$$

Therefore combining (13) and (14) we obtain

$$|||U^* \cdot f(T) \cdot U + f(0)(I - U^*U)||| \leqslant |||f(T)|||,$$

and so,

$$|||f(P^\lambda U P^{1-\lambda})||| \leqslant |||f(T)|||. \qquad \square$$

Corollary 3. *Let T, U, P, f, λ be the same as in Theorem* 1. *Then for* $0 < \rho$ *we have*

$$w_\rho\left(f(P^\lambda U P^{1-\lambda})\right) \leqslant w_\rho\left(f(T)\right). \tag{15}$$

In particular,

$$\|f(P^\lambda U P^{1-\lambda})\| \leqslant \|f(T)\| \quad \textit{and} \quad w\left(f(P^\lambda U P^{1-\lambda})\right) \leqslant w\left(f(T)\right). \tag{16}$$

Proof. For $0 < \rho \leqslant 2$, ρ-radius is a norm which satisfies conditions (7)–(10), and hence we can use Corollary 2(iii) to prove (15). In particular, if $\rho = 1$ and $\rho = 2$ then we have (16). We may assume $\rho > 2$. For $X \in \mathscr{B}(\mathscr{H})$ and $\rho > 1$ it is known (cf. [2]) that a necessary and sufficient condition of $w_\rho(X) \leqslant 1$ is $r(X) \leqslant 1$ and

$$\left\| \sum_{k=1}^{\infty} \frac{(\rho - 1)^k |z|^{k-1}}{\rho^k} X^k \right\| \leqslant \rho - 1 \quad \text{for } |z| \leqslant 1. \tag{17}$$

To prove (15) we need only check that if $w_\rho(f(T)) \leqslant 1$ then $w_\rho\left(f(P^\lambda U P^{1-\lambda})\right) \leqslant 1$. Let $w_\rho(f(T)) \leqslant 1$. Then by (16)

$$\begin{aligned} r\left(f(P^\lambda U P^{1-\lambda})\right) &= \lim_{n\to\infty} \|(f(P^\lambda U P^{1-\lambda}))^n\|^{1/n} \\ &\leqslant \lim_{n\to\infty} \|(f(T))^n\|^{1/n} = r(f(T)) \leqslant 1. \end{aligned}$$

Also since

$$\left\| \sum_{k=1}^{N} \frac{(\rho-1)^k |z|^{k-1}}{\rho^k} \left(f(P^\lambda U P^{1-\lambda})\right)^k \right\| \leqslant \left\| \sum_{k=1}^{N} \frac{(\rho-1)^k |z|^{k-1}}{\rho^k} (f(T))^k \right\|$$

for $|z| \leqslant 1$ and for any integer N by (16), taking a limit, we have

$$\begin{aligned} \left\| \sum_{k=1}^{\infty} \frac{(\rho-1)^k |z|^{k-1}}{\rho^k} \left(f(P^\lambda U P^{1-\lambda})\right)^k \right\| &\leqslant \left\| \sum_{k=1}^{\infty} \frac{(\rho-1)^k |z|^{k-1}}{\rho^k} (f(T))^k \right\| \\ &\leqslant \rho - 1 \quad \text{for } |z| \leqslant 1. \end{aligned}$$

So we can prove $w_\rho(f(P^\lambda U P^{1-\lambda})) \leqslant 1$. $\square$

Corollary 4 (cf. [5, Corollary 4.3]). *Let T, U, P, f, λ be the same as in Theorem* 1. *Then we have the following inclusion on the numerical range $W(\cdot)$:*

$$\overline{W\left(f(P^\lambda U P^{1-\lambda})\right)} \subset \overline{W(f(T))}.$$

Proof. In general, it is known that for $X \in \mathscr{B}(\mathscr{H})$ and each $1 \leqslant \rho \leqslant 2$,

$$\overline{W(X)} = \bigcap_{\mu\in\mathscr{C}} \{z; |z-\mu| \leqslant w_\rho(X-\mu)\}. \tag{18}$$

By Corollary 3, we can see that

$$w_\rho\left(f(P^\lambda U P^{1-\lambda}) - \mu\right) \leqslant w_\rho(f(T)-\mu).$$

Hence using (18) we have the conclusion. $\square$

Acknowledgements

The author would like to express his hearty thanks to T. Ando, M. Ito, H. Nakazato and T. Yamazaki for helpful discussions.

References

[1] A. Aluthge, On p-hyponormal operators for $0 < p < 1$, Integral Equations Operator Theory 13 (1990) 307–315.

[2] T. Ando, K. Okubo, Hölder type inequalities associated with operator radii and Schur products, Linear and Multilinear Algebra 43 (1997) 53–61.

[3] I.C. Gohberg, M.G. Krĕin, Introduction to the theory of linear nonselfadjoint operators, Amer. Math. Soc. 18 (1969) Translations of Mathematical Monographs.
[4] J.A.R. Holbrook, On the power-bounded operators of Sz.-Nagy and Foiaş, Acta Sci. Math. (Szeged) 29 (1968) 297–310.
[5] M. Ito, H. Nakazato, K. Okubo, T. Yamazaki, On generalized numerical range of the Aluthge transformation, Linear Algebra Appl., in press.
[6] Sz. Nagy, C. Foiaş, Harmonic analysis of operators on Hilbert space, North-Holland, Amsterdam, 1970.
[7] T. Yamazaki, On numerical range of the Aluthge transformation, Linear Algebra Appl. 341 (2002) 111–117.
[8] P.Y. Wu, Numerical range of Aluthge transform of operator, Linear Algebra Appl. 357 (2002) 295–298.